Rethinking what it means to understand: the case of combinatorial problem solving

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This paper argues for the need to address children's structural understanding in dealing with mathematical problems. In support of this argument, a study that investigated children's structural understanding of 2-D and 3-D combinatorial problems, via a range of thought-revealing tasks, is reported. The results raise a number of issues for further attention, including the discrepancy between children's accuracy and their structural understanding, and the lack of significant correlation between children's graphic and symbolic representations on most of the problems. Individual profiles of response highlight the importance of rethinking our interpretations of understanding.

With the increasingly widespread use of national and international comparative studies of school mathematics (e.g., Literacy and Numeracy Benchmarks; Third International Mathematics and Science Study [Stigler & Hiebert, 1997]), it is imperative that we do not fall into the trap of equating accuracy with understanding. The current testing frenzy is all the more reason to focus our attention on what it means to understand mathematical ideas, how we might best foster this understanding, and how we might assess for its presence. In this paper, I argue for the need to address children's *structural understanding* in dealing with mathematical problems. To illustrate my argument, I report on aspects of a study that implemented a range of tasks designed to assess this understanding in fifth-grade children's combinatorial problem solving (English, 1998a).

Defining Structural Understanding

This section addresses the nature of structural understanding, as used in the present study. First, it is necessary to define problem structure. The structure of a problem or problematic situation refers to the ways in which its mathematical ideas relate to each other, irrespective of the context in which the ideas are set (English, in press a; Novick, 1992). Structural understanding extends beyond a recognition of problem structure. For children to have developed a structural understanding of a given problem type (e.g., Cartesian products), they need to be able to:

1. explain the meaning of a problem;
2. represent the problem in different modes, including concrete, graphic and symbolic forms;
3. apply the processes of analogical reasoning to:
   (a) identify the structural elements of the problem;
   (b) detect the structural similarities and differences within, and between, related problems;
   (c) solve more complex cases of the given problem; and
   (d) pose new problems from the given problem. (English, 1998a)

Some might argue that aspects of (3), above, are an application of problem solving, rather than a fundamental component. On the contrary, a failure to apply the processes of analogical reasoning is one of the major causes of students' difficulties in dealing with mathematical problems (English & Halford, 1995; Holyoak & Thagard, 1995; Novick, 1995). Indeed, this was pointed out by Polya (1954) several decades ago.

Simply put, analogical reasoning entails understanding something new by analogy with something that is known, and is a fundamental process in children's mathematical learning and overall development (English, in press b). When applied to problem solving, reasoning by analogy first requires the solver to recognize and understand the structure of a given problem (known as the "source" or "base"), whether the problem be recalled from memory or supplied by an outside agent. Knowing the source structure can be of assistance in solving a new, related (target) problem, because the source structure can be mapped onto the structure of the target problem (Gentner & Gentner, 1983; Gentner, 1989; Holyoak & Thagard, 1995).

Despite the fact that reasoning by analogy contributes significantly to children's conceptual development during problem solving, it has received little attention from the mathematics education community (English, in press b, Holyoak & Koh, 1987; Novick, 1988, 1995; Silver, 1990). This is surprising, given that one of our major goals of mathematics education is for children to see the connections and relationships between mathematical ideas and to apply this understanding to the construction of new ideas and to the solution of new problems (English & Halford, 1995; Fuson, 1992; Hiebert, 1992; National Council of Teachers of Mathematics, 1989, 1991). It is thus imperative that children's problem experiences include a focus on the processes of analogical reasoning.

Combinatorial Problem Solving

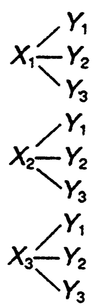
The present study investigated children's structural understanding of 2-D and 3-D combinatorial problems (i.e., X x Y [Cartesian products], and X x Y x Z). An analysis of the mathematical structure of these problems is presented in English, 1998a. In contrast to the focus placed on the other elementary problem types (e.g., Clark & Kamii, 1996; Kouba, 1989), combinatorial problems have received little research attention. This is a serious omission in the mathematics education literature, especially since they are one of the most difficult of the multiplication types for elementary school children (English, 1997; Harel & Confrey, 1994; Mulligan & Mitchelmore, 1997; Nesher, 1988; Outhred, 1996). Furthermore, combinatorics is a significant component of the curriculum, comprising a rich structure of significant mathematical principles that underlie several other areas such as counting, computation, and probability (English, 1993).

The existing literature has indicated that children employ a range of strategies in solving combinatorial problems (English, 1993, 1996a; Mulligan & Mitchelmore, 1997), that they frequently view the problems in terms of additive, combine problems (Nesher, 1992), and that, in general, they do not like dealing with problems of this nature, despite their colourful contexts (English, 1996b, 1998a). What seems to be lacking are studies that have undertaken a comprehensive analysis of children's structural understanding of these problems. To redress this situation, the present study employed a diverse range of "thought-revealing tasks" (as described in the methodology) that involved children in describing, explaining, representing, constructing, and justifying their ideas. (Lesh, Hoover, Hole, Kelly, & Post, in press; Lesh & Clarke, in press; Lester & Kroll, 1996).

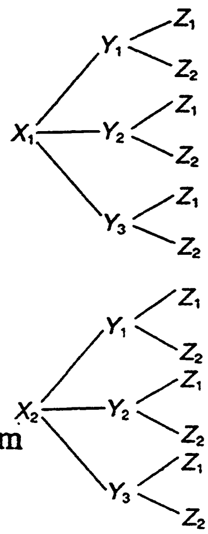
To provide some background for examining the children's responses, the next section reviews briefly some approaches to representing and solving combinatorial problems.

Approaches to Representing and Solving Combinatorial Problems

Research on children's strategies in solving combinatorial problems has not been prolific. Previous studies (e.g., English, 1993; English, 1996a) identified 10 different strategies children use in solving 2- and 3-D cases when presented in both concrete and word problem form. These strategies range from trial-and-error approaches to the efficient odometer procedure (cf. odometer of a vehicle). As indicated graphically in Figure 1 (a), this latter strategy involves holding constant an item from one set (set X) and systematically combining it with each of the items in the other set (set Y). When this procedure is applied to 3-D examples, two items are held constant simultaneously (from sets X and Y), while the third item (from set Z) is systematically varied (see Figure 1 [b]). It is beyond the scope of this paper to address all of the different solution strategies children have been observed to use. Of relevance here is the efficient odometer strategy, which was reflected in the children's representations, as mentioned in the results.



(a) 2-D problem



(b) 3-D problem

Figure 1 Graphic Representation of the Odometer Procedure

There is also a variety of graphic representations, including diagrams, sketches, and tables (Cunningham & Hubbold, 1992) that may be used both in representing and in solving combinatorial problems. As discussed later, these include systematic listing/drawings of combinations, hierarchies or branching structures ("tree diagrams;" Novick, 1996; see Figure 1), and matrices or rectangular arrays (Behr, Harel, Post, & Lesh, 1994; DeGuire, 1991; Diezmann, 1998; English & Halford, 1995; Novick, 1990). It has been argued that the ways in which children represent problem situations, whether these be in concrete, graphic, or symbolic form, are indicative of the ways in which they construct mathematical relationships (Davis, 1992; Diezmann, 1998; Hiebert & Wearne, 1992; Mulligan & Mitchelmore, 1996; Outhred, 1996). On the other hand, as Mulligan and Mitchelmore's (1996) study suggested, children may choose to structure their representations in a way that minimises cognitive load. Their study found that, although children's initial pictorial recordings reflected the semantic structure of a problem situation, some children restructured their representations to simplify calculation.

The important point here is that we cannot assume children understand combinatorial structures just because their problem representations might suggest this. Not until we observe children's responses across a range of problem situations, such as those described next, can we draw any conclusions.

METHOD

Participants

Thirty-two fifth-grade children from two schools participated in the study (mean age of 10.2 years). The schools were located in contrasting neighbourhoods of Brisbane. The children had been exposed to combinatorial problems in their mathematics curriculum.

Tasks and Procedures

The children were interviewed individually in two sessions, each of about 20 minutes duration. Whenever possible, the two sessions were conducted on consecutive days. Each child's responses were videotaped and transcribed for subsequent analysis. The problems presented in the two sessions appear in the appendix: Problems 1 - 4 are 2-D cases, while Problems 5 and 6 are 3-D examples. The following procedures were used.

Session 1

Each child was presented with Problems 1 and 2, which were placed on separate cards and read aloud for the child. When asked if they had experienced problems like these in class, all but four of the children said they had done so. The children were then directed to: (i) explain, in their own words, what each problem means, (ii) indicate how the two problems are similar to each other, and (iii) indicate how the two problems are different from each other. With respect to the latter, if the child simply referred to surface or contextual features, the question was asked, "Can you see any other way in which the problems are different from each other?"

The children were then directed to solve each problem, using the paper and colored pencils provided. The children were asked to show all of their working, and then to write a symbolic statement (number sentence) that represented their solution. The children were also asked whether they considered their problem to have been solved, and why. Finally, the children were invited to pose their own combinatorial problem, using one of the given examples as a base.

Session 2

The second session commenced with Problems 3 and 4 being presented along with two other problems that had different structures from Problems 3 and 4, but the same context. These problems were as follows: (a) In an activity class, Jenny and her friends made 6 blue skirts, 4 red skirts, and 6 white skirts to sell at the school fair. How many skirts did they make altogether? and (b) Tom has a big, hungry family. His family started the week with 5 loaves of white bread, 6 loaves of brown bread, 3 chickens, and 15 slices of corned beef. Towards the end of the week, the family had 1 loaf of white bread, 1 chicken, and 5 slices of corned beef left. How much of each item had his family eaten? The four problems were displayed randomly and the children were asked to select those problems that had similar structures (i.e., would be solved in a similar way) and to justify their selection.

Next, the children were directed to solve the two Cartesian product problems (Problems 3 and 4), using the paper and colored pencils provided. The children were asked to show all of their working, and then to write a symbolic statement (number sentence) that represented their solution. The children's solving of Problems 3 and 4 served as the source for the remaining activity, in which the 3-D target problems (Problems 5 and 6) were introduced. For these two target problems, the children were asked: (i) how the target problems are similar to, and different from, the two source problems (3 and 4), and (ii) how the source problems might help them solve the target problems. As before, the children were asked to solve each of the two target problems, to show all of their working, and to write a symbolic statement that represented the solution.

RESULTS

Children's responses to the present tasks have been examined in terms of the key components of structural understanding presented earlier, in addition to their ability to solve the problems. It is not possible to include all of the results within the present page limit; these will be presented at the conference.

Children's Accuracy

The children had few difficulties in solving the six problems. The percentages of children who solved each problem are as follows: 94% for each of Problems 1 and 2, 100% for Problem 3, 90% for Problem 4, 87% for Problem 5, and 77% for Problem 6. It is interesting that the children performed better on Problem 5 than 6, even though the latter is less complex computationally (2 x 2 x 2, compared with 3 x 2 x 2 for Problem 5).

Applying the Processes of Analogical Reasoning

Substantial evidence of the children's structural understanding of 2-D and 3-D combinatorial problems was gained from their ability to apply the processes of analogical reasoning. Included here are the children's: (i) identification of the similarities and differences between Problems 1 and 2, and likewise, for the remaining problems (as presented in Session 2); (ii) recognition of how source Problems 3 and 4 could assist in the solution of target Problems 5 and 6, and (iii) posing of new problems using Problems 1 and 2 as the base. These results are touched upon in the discussion.

Children's Graphic and Symbolic Representations

The nature of the children's graphic and symbolic representations provided further insights into the extent of their structural understanding.

Graphic Representations

There were five types of graphic representations identified in the children's responses (examples of these will be given in the conference presentation). In increasing order of abstractness, these representations are:

(i) <u>Random and/or partial listing of items, including the use of drawings or words</u>. These responses involved a random and/or partial listing of individual items (or combinations). Sometimes, such a response involved only a few single items being recorded randomly on the child's page.

(ii) <u>Systematic listings or drawings</u>. Responses in this category involved lists or drawings of combinations that were generated in a systematic manner. In the majority of cases, this listing reflected the use of an odometer procedure (as described previously), but occasionally a cyclic approach was evident (English, 1993). That is, in contrast to holding an item constant, the latter approach involves "cycling" through items in each set (e.g., *<u>white bread</u>/ham, <u>brown bread</u> /chicken, <u>multigrain bread</u>/corned beef, <u>white bread</u>/chicken, <u>brown bread</u>/corned beef* ....).

(iii) <u>System of Paths</u>. As described by Novick (1996), a system of paths does not have a formal structure, and does not necessarily have a unique starting or ending node, as in a branching structure or hierarchy (described next). The items in a system of paths have identical status, and the links between them can be associative, unidirectional, or bidirectional. There are no constraints on the linking of items (i.e., any item may be linked to any other item). When a child uses a system of paths in working the present problems, it is usually difficult to discern how the links between items have been made and also, whether a random or systematic procedure has been used in forming the links.

(iv) <u>A Branching or Hierarchical Structure.</u> This form of representation is commonly referred to as a "tree diagram," an example of which is given in Figure 1. Among the key features of this representation is the unidirectional nature of the links between items, that is, the path flows from a beginning node to an endpoint. Items at the same level cannot be linked, and neither can items in non-adjacent levels (Novick, 1996). Only a single path can enter each node but multiple paths can leave the node. Furthermore, as Novick pointed out, each node

includes implicit information about how to reach it. For the present problems, the odometer procedure is clearly reflected in the design of the branching structure.
(v) Quasi-matrix. As the name implies, this form of representation displays the basic features of a standard matrix, in that it has a rows-and-column format representing distinct sets of items. However, a quasi-matrix, as applied to this study, lacks some of the standard notations of a formal matrix. This may include an omission of the main grid lines or a failure to fully label them, and a failure to complete all cells. Nevertheless, these quasi-matrices display a factorial combination (Novick, 1996), where all the values of one variable (e.g., one type of bread) have the values of another in common (e.g., combined with a one type of filling to represent one type of sandwich). There is no linking of items within the one row or column (e.g., one type of bread cannot be linked with another type of bread).

No formal matrices were identified in the children's responses. This is not surprising, as children usually require specific instruction in using a standard matrix (Diezmann, 1998), and it is unlikely that such instruction would have been included in the children's fifth-grade curriculum. Table 1 displays the percentages of children who used each of the above representations in working each of the problems.

Table 1
Percentage of Children Using Each Form of Graphic Representation

	Form of Representation				
Problem	Random or partial listing of items	Systematic listing/drawing	System of paths	Branching structure	Quasi-matrix
1 (3 x 2)	37	19	9	13	13
2 (3 x 3)	13	22	6	22	22
3 (3 x 2)	13	25	9	16	22
4 (3 x 3)	16	22	13	13	22
5 (3 x 2 x 2)	9	28	19	19	16
6 (2 x 2 x 2)	16	31	16	19	9

Note. N = 32 The remaining children did not respond or gave a non-descript response.

Although it is difficult to discern clear trends in the children's use of graphic representation, there are some points worth noting. First, there appears some improvement in the sophistication of the children's representations, particularly between Problems 1 and 2. Second, very few children used a system of paths when dealing with the 2-D problems (although there was a small increase on the 3-D problems). Third, the use of systematic listing/drawing was favored overall, and, in particular, in dealing with the 3-D problems. Children painstakingly listed each of the combinations for Problems 5 and 6, instead of making use of a more efficient, albeit abstract, representation.

Also worth noting is the lack of consistency in the children's use of each type of graphic representation. Only nine children consistently used the one type of representation across all problems. Five of these children used systematic listing, three used a branching structure, and one randomly listed items. Of course, this lack of consistency may also be indicative of the children changing their graphic representation to suit the problem structure (e.g., adopting a different form of representation for the 3-D problems.). Ten children changed their form of representation when they moved to the 3-D problems, with this change not always being the most appropriate. Of these 10 children, three used two different representations in solving the 3-D problems.

Symbolic Representations

Children's symbolic representations of the problems included one-step and multi-step addition statements, one-step and multi-step multiplication, and the recording of an answer only. The proportions of children displaying each type of statement on each problem appear in Table 2.

Table 2
Percentage of Children who Recorded Each Type of Symbolic Statement

	Type of Symbolic Statement				
Problem	1-step addition	Multi-step addition	1-step multiplication	Multi-step multiplication	Answer only
1 (3 x 2)	22	16	37	3	6
2 (3 x 3)	3	12	59		9
3 (3 x 2)	10	3	50		9
4 (3 x 3)	3	25	59		6
5 (3 x 2 x 2)	6	22	53	6	9
6 (2 x 2 x 2)	25	12	37	9	6

Note. N=32 The remaining children did not respond.

Although multiplication statements were favored on each problem, there was, nevertheless, a considerable proportion of children who recorded addition statements, including multi-step addition. On Problems 1 and 6, in particular, the proportion of children who recorded addition statements was comparable to the proportion who gave multiplication statements. It is interesting that there was a noticeable decline in the use of multiplication between Problems 5 and 6. This reflects the previously cited decline in the children's accuracy between these problems.

Sixty-nine percent of the children were consistent in their symbolic representations across the problems. That is, if the children recorded an addition statement for Problem 1, they did so for all of the remaining problems; they did likewise if they commenced with a multiplication statement.

It is clear that the "primitive" repeated addition model of multiplication (Fischbein et al., 1985) still dominated the thinking of many of the children. Of particular concern, however, is the lack of multi-step multiplication statements for the 3-D problems (e.g., 2 x 2 x 2 = 8). The majority of children who wrote a multiplication statement for Problems 5 and 6 used only one step, such as, 4 x 3 = 12, and 4 x 2 = 8. This is an interesting finding and warrants further research (the work of Behr et al., 1994 would be valuable here).

One last finding of interest in this section is that there were only two significant correlations between children's graphic and symbolic representations, these occurring on Problems 3 and 5 ($r_s = .559$, $p < .01$ for Problem 3, and $r_s = .403$, $p < .05$ for Problem 5). For these problems, as children adopted the more abstract representations their symbolic statements changed from additive to multiplicative.

DISCUSSION

Children's responses to the problem tasks raise a number of issues. One of these pertains to the discrepancy between children's accuracy and the extent of their structural understanding. The majority of children could solve the problems, although a decline in accuracy occurred on Problem 6. Further research is needed to determine the reasons for this decline, whether it be the problem context, or the equivalent sets of items (2 x 2 x 2), or some other factors. In contrast to their accuracy across the problems, the children's responses suggest weaknesses in their structural understanding. For example, only a small proportion of the children could provide a comprehensive explanation of the structure of Problems 1 and 2. Although some children might have had difficulties in expressing their understanding in words, this did not seem to hinder them in justifying their solutions to these first two problems. Here, they showed a greater awareness of the 2-D combinatorial structure.

Nevertheless, this awareness did not extend to detailing the structural similarities and differences within, and between, the 2-D and 3-D problems. The cross-multiplication feature of the 2-D problems was rarely highlighted, and likewise, the subtle structural differences between the 2-D and 3-D problems remained undetected by most of the children. This could partly explain why the children rarely wrote a multistep multiplication statement for the 3-D problems. At least the children were generally able to recognize how their solving of the 2-D problems could assist them with the 3-D cases.

Another issue relates to the finding that the children's graphic and symbolic representations rarely correlated. This raises the question of the extent to which children's graphic and symbolic representations reflect their structural understanding of a given problem type (cf. Hiebert & Wearne's, 1992, argument). As Mulligan and Mitchelmore (1996) have noted, the multiplication notation children use in representing their solutions does not necessarily mean that they conceive of multiplication as a binary operation; they may simply be using it as a summary of their repeated addition process.

An analysis of individual cases highlighted the above issues. For example, Sally displayed a knowledge of the odometer procedure yet recorded addition statements only. Sarah, on the other hand, did not demonstrate this knowledge, yet recorded a multiplication statement for each problem. Kate and Siobhan used sophisticated graphic representations and also recorded multiplication statements. However, neither girl could explain the meaning of Problems 1 and 2, and neither could give a detailed account of their structural similarities. On the other hand, they were more adept at detecting the structural differences among the different problems. Kate was able to pose a solvable problem, while Siobhan could not.

This study has demonstrated the need to implement a broader range of thought-revealing tasks both in promoting and assessing children's development of structural understanding. We cannot assume that children fully understand problems of a given type just because they can solve them or can represent them in some acceptable format. This paper has attempted to illustrate the importance of looking beyond these skills to address the components of structural understanding that have been advanced. We should then have greater confidence in our children's attainments on comparative tests of school mathematics.

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Appendix

Problems Presented in the Two Sessions

Session 1

Problem 1
Jenny has a blue skirt, a red skirt, and a yellow skirt. With these, she can wear a white T-shirt or a green T-shirt. How many different outfits can she make?

Problem 2
Tom is making sandwiches for a picnic. He has white bread, brown bread, and multigrain bread. He can fill these with ham, chicken, or corned beef. How many different sandwiches can he make if each sandwich has one type of bread and one type of filling?

Session 2

Problem 3 (source problem)
Mrs. Jones is trying to decide on a flower arrangement. She can choose from roses, carnations, and daffodils. She can use a tall vase or a short vase. How many different arrangements could she make, if she were to use only one type of flower and one type of vase?

Problem 4 (source problem)
Mark is making crazy animals. For the animals' heads, he can choose from a rooster, a monkey, and a duck. For their bodies, he can choose from a donkey, a rabbit, and an elephant. How many different crazy animals can he make?

Problem 5 (target problem)
Marina is making boxes of greeting cards to sell at the fair. She has blue paper, red paper, and green paper. She also has striped ribbon, and spotted ribbon. For the lettering, she can use gold ink or silver ink. Each box will have different greeting cards. How many different greeting cards will she put in each box, if each card has one colored paper, one ribbon, and lettering of one color?

Target Problem 6 (target problem)
Mrs. Jones needs a new car but cannot decide what to buy. Here are her choices. She can choose from a 2-door or a 4-door car. It can have luxury seat covers or standard seat covers. It can also have metallic paint or regular paint. How many different choices has she?

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